VIBRATIONS OF AN ORTHOTROPIC HALF-PLANE WITH A CAVITY

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Recently the vibrations of semi-bounded isotropic bodies with cavities have been studied in detail by using superposition for canonically shaped cavities. Analyzing a wave field with anisotropy and noncanonically shaped defects requires completely different methods, including the method of limiting integral equations. The method of limiting elements, which was developed in recent years is based on knowing the fundamental solutions for an orthotropic medium. Here the methods of limiting integral equations and limiting elements are applied to orthotropic media based on a fundamental solution which we constructed in the form of singular integrals along a contour in the complex plane; here such a representation allows effective construction of the matrix coefficients of the system in the form of singular integrals with the simplest approximation of the unknowns in the element. It also allows the wave field to be examined at any point in the medium.

1. We examine the problem of establishing the vibrations of an elastic orthotropic half-plane $x_3 < 0$, which is weakened by a void with a smooth boundary ℓ. We assume that the vibrations of the half-plane are caused by a normal load applied at the boundary $x_3 = 0$. After the time multiplier is removed, the equations of motion and the boundary conditions have the form

$$\sigma_{ij,j}+\rho\omega^2 u_i=0 \quad (i=1,\ 3,\ j=1,\ 3); \tag{1.1}$$

$$\sigma_{11}=C_{11}\varepsilon_{11}+C_{13}\varepsilon_{33},\ \sigma_{33}=C_{13}\varepsilon_{11}+C_{33}\varepsilon_{33},\ \sigma_{13}=2C_{55}\varepsilon_{13}, \tag{1.2}$$

for $x_3 = 0, \sigma_{13} = 0,\ \sigma_{33} = -p(x_1)$

$$p(x_1)=\begin{cases}0, & x_1\notin l_1,\\ p(x_1), & x_1\in l_1,\end{cases}\quad \sigma_{ij}n_j|_l=0\ ,$$

where n_j are the components of the unit vector which point outwards into the elastic medium, normal to the curve ℓ. The radiation conditions, which are formulated using the principle of limiting contours [1], complete the problem specification.

A representation of the fundamental solution, formulated [2] for an orthotropic half-plane, can be used to reduce the boundary problem to a system of limiting integral equations. The fundamental solution, which satisfies homogeneous boundary conditions for $x_3 = 0$, $\sigma_{13}^{(m)} = 0$, $\sigma_{33}^{(m)} = 0$ ($m = 1, 3$) is constructed using the Fourier transform:

$$U_k^{(m)}=\mathring{U}_k^{(m)}+S_k^{(m)}\quad (k,\ m=1,\ 3),$$

$$\mathring{U}_k^{(m)}(x_1,\ x_3,\ \xi_1,\ \xi_3)=(4\pi C_{55})^{-1}r_{km}(x_3,\ \xi_3)\int_\sigma \exp[ik\beta(\xi_1-x_1)]\times$$

$$\times\sum_{j=1}^{2}(-1)^{j-1}\varphi_k^{(m)}(\lambda_j)\,\mu^{-1}(\lambda_j)\exp(-k\lambda_j|\xi_3-x_3|)\,d\beta; \tag{1.3}$$

$$S_k^{(m)}(x_1,\ x_3,\ \xi_1,\ \xi_3)=(4\pi C_{55})^{-1}\int_\sigma \exp[ik\beta(\xi_1-x_1)]\sum_{r,p=1}^{2}Q_{krp}^{(m)}(\beta)\,\Delta^{-1}(\beta)\,\mu^{-1}(\lambda_r)\exp[k(\lambda_r\xi_3+\lambda_p x_3)]\,d\beta. \tag{1.4}$$

Here

$$\begin{gathered} r_{km}(x_3,\ \xi_3)=i(1-\delta_{km})\,\mathrm{sign}(\xi_3-x_3)+\delta_{km},\\ \lambda_j=\lambda_j(\beta)=[(2\gamma_5)^{-1}(-b(\beta)\pm d^{1/2}(\beta))]^{1/2},\\ b(\beta)=(\gamma_7^2+2\gamma_5\gamma_7-\gamma_1)\beta^2+1+\gamma_5,\quad c(\beta)=(\gamma_1\beta^2-1)(\gamma_5\beta^2-1),\\ d(\beta)=b^2(\beta)-4\gamma_5 c(\beta),\ \mu(\lambda)=(\lambda_1^2-\lambda_2^2)\lambda. \end{gathered} \tag{1.5}$$

The following parameters are introduced

$$\gamma_1=C_{11}/C_{33},\ \gamma_5=C_{55}/C_{33},\ \gamma_7=C_{13}/C_{33},\ k^2=\rho\omega^2/C_{33},$$

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$$\varphi_1^{(1)}(\lambda)=\lambda^2-\gamma_5\beta^2+1,\ \ \varphi_1^{(3)}(\lambda)=\varphi_3^{(1)}(\lambda)=(\gamma_5+\gamma_7)\lambda\beta,$$
$$\varphi_3^{(3)}(\lambda)=\gamma_5\lambda^2-\gamma_1\beta^2+1,$$
$$Q_{11j}^{(m)}(\beta)=K_{1j}^{(m)}(\beta)\varphi_1^{(3)}(\lambda_j)(i\beta)^{\varkappa_m}\quad(\varkappa_1=0,\ \varkappa_3=1),$$
$$Q_{31j}^{(m)}(\beta)=K_{1j}^{(m)}(\beta)R_{3j}(\beta)\quad(j=1,2),$$

and $Q_{k22}^{(m)}(\beta)$ and $Q_{k21}^{(m)}(\beta)$ are obtained from $Q_{k11}^{(m)}(\beta)$ and $Q_{k12}^{(m)}(\beta)$, respectively, by substituting $\lambda_1 \leftrightarrow \lambda_2$;

$$K_{11}^{(1)}(\beta)=\lambda_1\lambda_2A(\lambda_1)P(\lambda_2)-\beta^2Q(\lambda_2)B(\lambda_1),$$
$$K_{12}^{(1)}(\beta)=\beta^2Q(\lambda_1)B(\lambda_1)-A(\lambda_1)\lambda_1^2P(\lambda_1),$$
$$K_{11}^{(3)}(\beta)=-\beta(\lambda_1P(\lambda_1)Q(\lambda_2)+\lambda_2P(\lambda_2)Q(\lambda_1)),$$
$$K_{12}^{(3)}(\beta)=2\lambda_1P(\lambda_1)Q(\lambda_1),\quad A(\lambda)=\lambda^2+\gamma_7\beta^2+1,\tag{1.6}$$
$$B(\lambda)=\gamma_5\lambda^2+\gamma_5\gamma_7\beta^2-\gamma_7,\quad P(\lambda)=\left(\gamma_7^2+\gamma_5\gamma_7-\gamma_1\right)\beta^2+\gamma_5\lambda^2+1,$$
$$Q(\lambda)=\gamma_1\beta^2+\gamma_7\lambda^2-1,\quad R_{3j}=\gamma_5\lambda_j^2-\gamma_1\beta^2+1,$$
$$\Delta(\beta)=\beta(\gamma_5+\gamma_7)(\lambda_2-\lambda_1)\left[\lambda_1\lambda_2\left(\beta^2\left(\gamma_1-\gamma_7^2\right)-1\right)-(\gamma_1\beta^2-1)\right].$$

The integrands in (1.4)-(1.6) have two pairs of branch points on the real axis: $\beta = \pm\gamma_1^{-1/2}$ and $\beta = \pm\gamma_5^{-1/2}$, and also two pairs which are determined from the equation $d(\beta) = 0$. Depending on the combination of the elastic constants, we note that this equation can have real, purely imaginary, and complex roots [3]. Moreover, the function $\Delta(\beta)$ has a zero on the real axis, which corresponds to the Rayleigh wave for an orthotropic half-plane [4]. The contour σ coincides with the real axis everywhere except for real poles and branch points, which it circumvents according to the principle of limiting contours as follows: positive poles bend downwards and negative ones bend upwards.

2. Based on the dynamic reciprocity theorem [5] and an analysis of the limiting values of the fundamental solution [2], the problem is reduced to a system of limiting integral equations relative to the displacements on the boundary of the cavity:

$$\frac{1}{2}u_m(y)+\int\sigma_{kj}^{(m)}(x,y)n_j(x)u_k(x)\,dl_x=u_m^{st}(y),\quad x=(x_1,x_3),\,y=(y_1,y_3)\in l.\tag{2.1}$$

We note that the integral in (2.1) is a principle value integral in the Cauchy sense and $u_m^{st}(y)=-\int_{l_1}p(x_1)U_3^{(m)}(x_1,0,y)\,dx_1$ is the standard displacement in the medium without a cavity.

Here $\sigma_{kj}^{(m)}(x, y)$ is found according to the defining equations (1.2), into which we substitute $\varepsilon_{ij}^{(m)}=(1/2)\left(U_{i,j}^{(m)}+U_{j,i}^{(m)}\right)$ instead of ε_{ij}, and $U_j^{(m)}$ in turn has the form of (1.3).

Integral equations of the type (2.1) are analyzed efficiently by the boundary element method, according to which the cavity boundary ℓ is approximated by a piecewise curve ℓ_0 of N elements. Let (x_{1p}^-, x_{3p}^-) be the coordinates of the start of the p-th element and (x_{1p}^+, x_{3p}^+) be the coordinates of its end; then the nodes will be points with coordinates (y_{1p}, y_{3p}), where

$$y_{kp}=\left(x_{kp}^-+x_{kp}^+\right)/2\quad(k=1,3,\ p=1,2,\ldots,N).$$

During the discretization we will assume that the components of the displacement vector u_m are constant on the element p and equal to the displacements of the corresponding node:

$$u_m(y_{1p},\ y_{3p})=u_{mp}.\tag{2.2}$$

By using the collocation method, we require that the system (2.1) be satisfied for the nodal points (y_{1q}, y_{3q}), where $q = 1, 2,\ldots,N$. As a result, we come to a linear algebraic system for 2N unknown nodal displacements:

$$u_{mp}/2+\sum_{r=1}^{N}a_{mprj}u_{jr}=g_{mp}\quad(m,\,j=1,3,\ p=1,2,\ldots,N).\tag{2.3}$$

Here

$$a_{mprj}=\int_{l_r}\sigma_{jk}^{(m)}(y_{1p},y_{3p},x_1,x_3)\,n_k(x_1,x_3)\,dl_x,\qquad g_{mp}=u_m^{st}(y_p)\quad(m,j=1,3,\ p=1,2,\ldots,N).\tag{2.4}$$

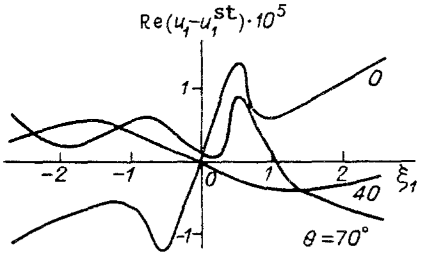

Fig. 1

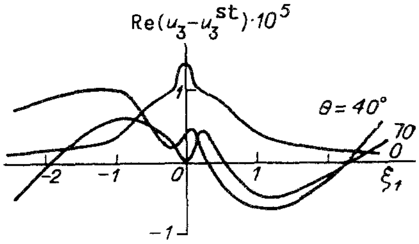

Fig. 2

We note that after integration over an element in (2.4), amprj is represented in the form of singular integrals over the contour σ, as was done in [6]. Due to the complexity of the resultant expressions, these integrals are not produced here. Once we solve the system (2.3) and find the value of the nodal displacements, we determine the wave field at the boundary of the half-plane:

$$u_m(\xi_1, 0) = -\sum_{r=1}^{N} a_{mj}(\xi_1, 0, y_{1r}, y_{3r})\, u_{jr} + g_m(\xi_1, 0),$$
$$a_{mj}(\xi_1, 0, y_{1r}, y_{3r}) = \int_{-1}^{1} \left(-\sigma_{j1}^{(m)}(\xi_1, 0, y_{1r} + \beta_{1r}t, y_{3r} + \beta_{3r}t)\beta_{3r} + \right. \tag{2.5}$$
$$\left. + \sigma_{j3}^{(m)}(\xi_1, 0, y_{1r} + \beta_{1r}t, y_{3r} + \beta_{3r}t)\beta_{1r}\right) dt.$$

A Fortran program was written for the ES-1055 computer to calculate the wave field at the boundary of the half-plane, to determine the nodal displacements at the boundary of the cavity, and then to calculate the wave field at the boundary of the half-plane from Eq. (2.5) for a given number of decomposition points of the cavity. Here the integral over the contour σ is calculated using Gauss quadrature. In the numerical calculations, the vibration source was taken to be a point force applied at the origin of the coordinates; i.e., $p(x_1) = p\delta(x_1)$. The region ℓ was taken to be an elipsis with semiaxes d_1 and d_3, with center coordinates x_{10} and x_{30}, which were rotated about the axis Ox_1 by an angle $\theta_i = \pi i/18$, $i = 1, 2, \ldots, 9$. The calculations were done for (austenite steel):

$$C_{11} = 26{,}27 \cdot 10^{10}\ \text{N/m}^2,\quad C_{13} = 14{,}5 \cdot 10^{10}\ \text{N/m}^2,$$
$$C_{33} = 21{,}6 \cdot 10^{10}\ \text{N/m}^2,\quad C_{55} = 12{,}9 \cdot 10^{10}\ \text{N/m}^2$$

and geometric parameters

$$d_1 = 0{,}2,\ d_3 = 0{,}1,\ x_{10} = 0,\ x_{30} = -0{,}5.$$

Eight and sixteen elements were used to approximate the cavity ℓ. We note that the relative error in the displacements at the boundary of the half-plane did not exceed 5% for 8 and 16 elements. Moreover, when the cavity was approximated by an inscribed and circumscribed octagon, with $k \leqslant 3$, the error in the calculated wave field at the surface did not exceed 6%. As the wave number k was increased ($k \geqslant 5$), the efficiency of this algorithm was reduced, due to the very simple approximation of the type (2.2) for the unknowns on the element. However, the discretization method was retained, even when the unknown functions were approximated by linear and quadratic functions on the element, and the coefficients of the system (2.4) were also expressed in the form of singular integrals over the contour σ. Figures 1 and 2 show $\text{Re}(u_1 - u_1^{st})$ and $\text{Re}(u_3 - u_3^{st})$ as functions of the angle θ. These functions can be used as initial information for solving the inverse problem of determining the shape of a defect from the reflected field.

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PARAMETRIC INSTABILITY IN THE OSCILLATIONS OF A BODY MOVING UNIFORMLY IN A PERIODICALLY INHOMOGENEOUS ELASTIC SYSTEM

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Transitional radiation of various types occurs in the uniform and rectilinear motion of a perturbation source in an inhomogeneous medium. In [1, 2], there is a survey of such radiation for electromagnetic and acoustic waves. In [3], the radiation was examined for elastic waves arising in the uniform motion of a mechanical object in an inhomogeneous elastic system. Features of the radiation in such a system are related to the interaction between the radiation and the oscillations of the object. Here we consider parametric object oscillation excitation during emission.

When a perturbing source moves in a periodically inhomogeneous medium, the radiation has a discrete spectrum in the steady state [1]. In a reference system coupled to the moving source, the spectrum is equidistant. The object moving uniformly in a periodically inhomogeneous elastic system is subject to a transverse force equivalent to the reaction of a spring with periodically varying rigidity. That situation naturally leads to parametric object oscillation [4], which is demonstrated here. It is necessary to examine such interaction for example in relation to the requirements for high-speed railroad transportation. A train moving over rails under certain conditions may begin to show a galloping motion, and here we show that the parameter region where it occurs expands as the speed increases.

1. We consider the uniform motion $z = vt$ of a body with mass m along an unbounded string whose tension and density per unit length are correspondingly N and ρ, and which lies on a periodically inhomogeneous elastic base. The rigidity of the base is described by

$$k(z) = k_0(1 + \mu\cos(2\pi z/d)),$$

in which k_0 is the mean rigidity, d the period of the inhomogeneity, and $\mu \ll 1$ a dimensionless small parameter.

A description of the self-consistent motion of the body and string is [5]

$$U_{tt} - U_{xx} + U(1 + \mu\cos(\varkappa x)) = 0,\ U(\alpha t,\ t) = y(t),$$
$$(1-\alpha^2)[U_x]_{x=\alpha t} = M\ddot{y}(t),\ [U]_{x=\alpha t} = 0,\ U \to 0 \text{ for } x \to \pm\infty. \tag{1.1}$$

Here U(x, t) is the transverse deviation of the string, $x = zh/c$ and $t = h\tau$ ($c^2 = N/\rho$, $h^2 = k_0/\rho$) are the dimensionless coordinate and time, y(t) the transverse coordinate of the body, with $M = mh/c\rho$ and $\alpha = v/c$ (with $\alpha < 1$ subsequently) the dimensionless mass and longitudinal velocity, and $\varkappa = 2\pi c/dh$. The square brackets denote the differences between the values of the expressions in them to the right and left of the given x.

We seek the solution to (1.1) as

$$U = U^0 + \mu U^1 + \ldots,\ y = y^0 + \mu y^1 + \ldots \tag{1.2}$$

2. In the zeroth approximation ($\mu = 0$), (1.2) represents the motion of a body on a string lying on a homogeneous elastic base:

$$U^0_{tt} - U^0_{xx} + U^0 = 0,\quad U^0(\alpha t,\ t) = y^0(t),$$
$$(1-\alpha^2)[U^0_x]_{x=\alpha t} = M\ddot{y}^0(t),\ [U^0]_{x=\alpha t} = 0,\ U^0 \to 0 \text{ for } x \to \pm\infty. \tag{2.1}$$

As the solution to (2.1), one naturally takes a function describing the oscillation of the body–string system for $t \to \infty$. We first determine the oscillation frequency for $t \to \infty$.

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